Existence theorems for elliptic equations in unbounded domains

Armen Beklaryan[1]

[1] *Lomonosov Moscow State University, Moscow, Russia;*
`beklaryan@hotmail.com`

We consider the first boundary value problem for elliptic systems defined in unbounded domains, which solutions satisfy the condition of finiteness of the Dirichlet integral also called the energy integral

$$\int\limits_{\Omega} |\nabla u|^2 dx < \infty.$$

Basic concepts

Let Ω is an arbitrary open set in $\mathbb{R}^n$. As is usual, by $W^1_{2,\,loc}(\Omega)$ we denote the space of functions which are locally Sobolev, i.e.

$$W^1_{2,\,loc}(\Omega) = \{f : f \in W^1_2(\Omega \cap B^x_\rho),\, \forall\, \rho > 0\,,\, \forall\, x \in \mathbb{R}^n\}\,,$$

where B^x_ρ – open ball with center at point x and with radius ρ. If $x = 0$ then we will write B_ρ. We will denote by $\overset{\circ}{W}{}^1_{2,\,loc}(\Omega)$ set of functions from $W^1_{2,\,loc}(\mathbb{R}^n)$, which is the closure of $C^\infty_0(\Omega)$ in the system of seminorms $\|u\|_{W^1_2(\mathcal{K})}$, where $\mathcal{K} \subset \mathbb{R}^n$ are various compacts. Let denote by $L^1_2(\Omega)$ a space of generalized functions in Ω, which first derivatives belong to $L_2(\Omega)$ [4], in other words

$$L^1_2(\Omega) = \{f \in \mathcal{D}^{'}(\Omega) : \int\limits_{\Omega} |\nabla f|^{\,2} dx < \infty\}.$$

Let $\omega \subseteq \mathbb{R}^n$ is an open set, $\mathcal{K} \subset \omega$ is a compact. We will denote by $\Phi_\varphi(\mathcal{K}, \omega)$ the set of functions $\psi \in C^\infty_0(\omega)$ such that $\psi = \varphi$ in the neighborhood of $\mathcal{K}$, or in other words $\psi - \varphi \in \overset{\circ}{W}{}^1_{2,\,loc}(\mathbb{R}^n \setminus \mathcal{K})$.

Let's define a capacitance of a compact $\mathcal{K}$ relative to the set ω [4]:

$$\operatorname{cap}_{\,\varphi}(\mathcal{K}, \omega) = \inf_{\psi \in\, \Phi_\varphi(\mathcal{K},\,\omega)} \int\limits_{\omega} |\nabla \psi|^{\,2} dx\,.$$

The capacitance of arbitrary closed set $E \subset \omega$ in $\mathbb{R}^n$ is defined by the formula $\operatorname{cap}_\varphi(E,\omega) = \sup_{\mathcal{K}\subset E} \operatorname{cap}_\varphi(\mathcal{K},\omega)$. If $\omega = \mathbb{R}^n$, then instead of $\operatorname{cap}_\varphi(E,\mathbb{R}^n)$ we will write $\operatorname{cap}_\varphi(E)$.

Problem statement

Let L is a divergent operator

$$L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right),$$

where a_{ij} are bounded measurable functions in $\mathbb{R}^n$ satisfying condition

$$\gamma|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\, \xi_i\, \xi_j\,, \quad \xi \in \mathbb{R}^n, \gamma > 0\,.$$

The solution of the Dirichlet problem

$$\begin{cases} Lu & = 0 \text{ in } \Omega \\ u|_{\partial\Omega} & = \varphi, \end{cases} \tag{1}$$

where $\varphi \in W^1_{2,\,loc}(\mathbb{R}^n)$, is a function $u \in W^1_{2,\,loc}(\Omega)$ such that:

1) $u - \varphi \in \overset{\circ}{W}{}^1_{2,\,loc}(\Omega)$, i.e. $(u - \varphi)\mu \in \overset{\circ}{W}{}^1_2(\Omega)$ for any function $\mu \in C_0^\infty(\mathbb{R}^n)$;

2) function u has bounded Dirichlet integral

$$\int\limits_\Omega |\nabla u|^2 dx < \infty\,;$$

3)

$$\int\limits_\Omega \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial \psi}{\partial x_i}\, dx = 0$$

for any function $\psi \in C_0^\infty(\Omega)$.

Basic results

Theorem 1. *Let's* $cap_{\varphi - c}(\mathbb{R}^n \setminus \Omega) < \infty$ *for some constant* $c \in \mathbb{R}^n$*. Then the problem (1) has a solution.*
Theorem 2. *Let the problem (1) has a solution and it is true that*

$$\int\limits_{\mathbb{R}^n \setminus \Omega} |\nabla \varphi|^2 dx < \infty \,.$$

Then there is such constant $c \in \mathbb{R}^n$*, that* $cap_{\varphi - c}(\mathbb{R}^n \setminus \Omega) < \infty$*.*
Theorem 3. *Let* $n \geq 3$*. Then* $cap_{\varphi - c}(\mathbb{R}^n \setminus \Omega) < \infty$ *if and only if*

$$\sum_{k=N}^{\infty} cap_{\varphi - c}((\overline{B}_{2^{k+1}} \setminus B_{2^{k-1}}) \cap (\mathbb{R}^n \setminus \Omega), B_{2^{k+2}} \setminus \overline{B}_{2^{k-2}}) < \infty$$

for some $N \in \mathbb{N}$*.*

Particular cases

Let consider the space $\mathbb{R}^n$ with a set of coordinates $(x_1, x_2, \ldots, x_n)$ and let $\varphi_\alpha = (1+|\,x_1|)^{\,\alpha}$. Domain $\Omega_{1,i}$ is upper half-plane relative to x_i, where $i \neq 1$, in other words $\Omega_{1,i} = \{(x_1, x_2, \ldots, x_n) | x_i \geq 0, i \neq 1\}$. Domain Ω_2 is the outer part of the space formed by surface of revolution relative to x_1 of the curve from Fig.1.

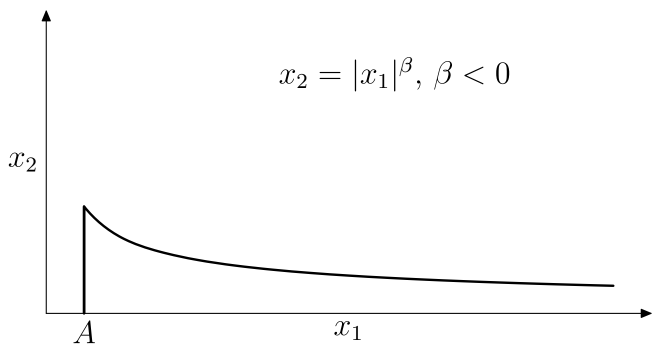

Fig. 1: Domain Ω_2

Corollary 1. *Let $n \geq 2$. Then for the domain $\Omega_{1,i}$ and for bounded function φ_α the existence of solutions of the problem (1) is equivalent to either an inequality $\alpha < -\frac{1}{2}$ or $\alpha = 0$.*

Corollary 2. *Let $n \geq 3$. Then for the domain Ω_2 and for bounded function φ_α the existence of solutions of the problem (1) is equivalent to either an inequality $\alpha < -\frac{1+\beta(n-3)}{2}$ or $\alpha = 0$.*

REFERENCES

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